quadrature schemes may be used in effecting the numerical integrations. In this sense, specific discrete element "condensation" techniques represent only one of many possible ways for carrying out the required process of integration and summation in the Rayleigh-Ritz method.

To further illustrate the alternative ways in which Rayleigh's principle may be applied and the fact that the use of a vibration mode determined from a first iteration does not necessarily lead to best results, the numerical example given by Fried[3] will be considered from the simpler (and more traditional) point of view of the dynamicist. The Rayleigh quotient may be formed from Fried's Eq. (6) by multiplying the first row of the matrix equation by x_1, the second by x_2, etc., and summing the results to give

$$\lambda = \frac{x_1^2+(x_2-x_1)^2+(x_3-x_2)^2+(x_4-x_3)^2+(x_5-x_4)^2+x_5^2}{x_1^2+x_2^2+x_3^2+x_4^2+x_5^2} \tag{1}$$

Apart from physical constants which are absorbed in the definition of λ, this may be viewed as Rayleigh's quotient for the transverse vibration of a massless string under constant tension with five equal masses equally spaced along the string. In this case the numerator represents the strain energy and the denominator represents the kinetic energy divided by frequency squared. In the traditional approach of the dynamicist, a reasonable (and typical) assumption for the shape of the lowest symmetric mode of this system would be a parabola (as in Appendix 1 of Ref. 4). Thus

$$x_1 = 5, \quad x_2 = 8, \quad x_3 = 9, \quad x_4 = 8, \quad x_5 = 5$$

Substituting these values into Eq. (1) gives

$$\lambda_1 = 70/259 = 0.2703$$

This compares to Fried's result for the singly iterated mode, $\lambda_1 = 0.2860$, for the doubly iterated mode, $\lambda_1 = 0.2680$, and the exact value, $\lambda_1 = 0.2679$. The accuracy of the result obtained here by application of the primitive Rayleigh method is thus much better than the result obtained by Fried using a singly iterated mode and differs from the exact solution by less than 1%.

For the antisymmetrical case, applying a similar assumed parabola to each half of the system leads to $\lambda_2 = 1.000$ which corresponds to the exact solution. By contrast, Fried obtained $\lambda_2 = 1.200$ for his singly iterated mode and $\lambda_2 = 1.024$ for his doubly iterated mode. Fried's results are a consequence of the fact that his assumed antisymmetric loading for the first iteration destroyed the subsymmetry of the first antisymmetric vibration mode in each half of the system.

This example is not, of course, typical of all problems to which Rayleigh's principle might be applied. (For instance, in the bending vibration of cantilever beams of variable properties use of the first iteration for the mode shape in application of Rayleigh's principle to determination of the lowest natural frequency is often a most satisfactory procedure.) It does, however, emphasize that to obtain best results with minimum effort from Rayleigh's principle, it is necessary for the analyst to exercise a degree of judgment based on the geometrical, structural, and dynamic properties of the system under consideration and to choose analytical techniques accordingly.

References

[1] Householder, A. S., *Principles of Numerical Analysis*, McGraw-Hill, New York, 1953.

[2] Wilkenson, J. H., *The Algebraic Eigenvalue Problem*, Oxford Univ. Press, New York, 1965.

[3] Fried, I., "Condensation of Finite Element Eigenproblems," *AIAA Journal*, Vol. 10, No. 11, Nov. 1972, pp. 1529–30.

[4] Temple, G. and Bickley, W. G., *Rayleigh's Principle and its Application to Engineering*, Oxford Univ. Press, New York, 1933, republished by Dover, New York, 1956.

[5] Frazer, R. A., Duncan, W. J., and Collar, A. R., *Elementary Matrices*, Cambridge Univ. Press, New York, 1938, pp. 308–331.

[6] Morse, P. M. and Feshbach, M., *Methods of Theoretical Physics*, Vol. II, McGraw-Hill, New York, 1953, pp. 1026–1030.

[7] Collatz, L., *Eigenwertproblem und Ihre Numerische Behandling*, Chelsea, New York, 1948, pp. 253–257.

[8] Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., *Aeroelasticity*, Addison-Wesley, Reading, Mass., 1955, pp. 132–172.

[9] Rauscher, M., "Station Functions and Air Density Variations in Flutter Analysis," *Journal of Aerospace Sciences*, Vol. 16, No. 6, June 1949, pp. 345–353.

[10] Timoshenko, S., *Vibration Problems in Engineering*, D. Van Nostrand, New York, 1937, p. 380.

Periodic Solutions of Gravity Oriented Axisymmetric Systems

V. J. MODI* AND K. KUMAR†

The University of British Columbia, Vancouver, British Columbia, Canada

1. Introduction

THE periodic solutions of the planar librational motion for satellites in an eccentric orbit were first investigated by Zlatousov et al.[1] More recently, the analyses by Modi and associates[2–5] clearly established their usefulness in providing valuable information concerning the formation of a stability region and the critical eccentricity beyond which no stable motion is possible. Interestingly, at the critical eccentricity, the only available solution is a periodic one. Thus, periodic solutions play an important role in the attitude dynamics study.

This Note studies initial conditions leading to periodic motion for gravity gradient stabilized satellites using an extension of the Krylov and Bogoliubov method (variation of parameter procedure)[6] as suggested by Butenin[7] with certain modifications.[8,9] The validity of the results, obtained over a wide range of system parameters, is assessed through comparison with the response as given by numerical integration of the exact equations of motion. The effect of solar radiation pressure (the predominant environmental force at higher attitudes) is also included to emphasize its significance. The procedure represents a simple yet effective model for response evaluation during the preliminary design stage.

2. Equations of Motion and Analysis

Using the Lagrangian formulation, the governing equations for pitch (ψ), roll (β) and yaw (λ) librations (Fig. 1) for an arbitrarily shaped axisymmetric, gravity oriented satellite can be written as[10]

$$(1+e\cos\theta)\psi'' - 2e(1+\psi')\sin\theta - 2(1+e\cos\theta)(1+\psi')\beta'\tan\beta + 3K_i\sin\psi\cos\psi = Q_\psi \tag{1a}$$

$$(1+e\cos\theta)\beta'' - 2e\beta'\sin\theta + [(1+e\cos\theta)(1+\psi')^2 + 3K_i\cos^2\psi] * \sin\beta\cos\beta = Q_\beta \tag{1b}$$

$$\lambda' - (1+\psi')\sin\beta = 0 \tag{1c}$$

where e is the orbital eccentricity, θ is the position of the satellite as measured from pericenter, K_i is the inertia parameter, $(I_{xx}-I_{zz})/I_{yy}$, I_{jj} is the principal moments of inertia about j-axis ($j = x, y, z$), Q_ψ, Q_β are the generalized forces, complicated functions of geometry of the satellite, its orbit, mass distribution and spatial orientation, and primes denote differentiation with respect to θ.

Received March 8, 1973; revision received August 27, 1973. The investigation was supported by the National Research Council of Canada, Grant Number A-2181.

Index category: Spacecraft Attitude Dynamics and Control.

* Professor, Department of Mechanical Engineering.

† Research Associate.

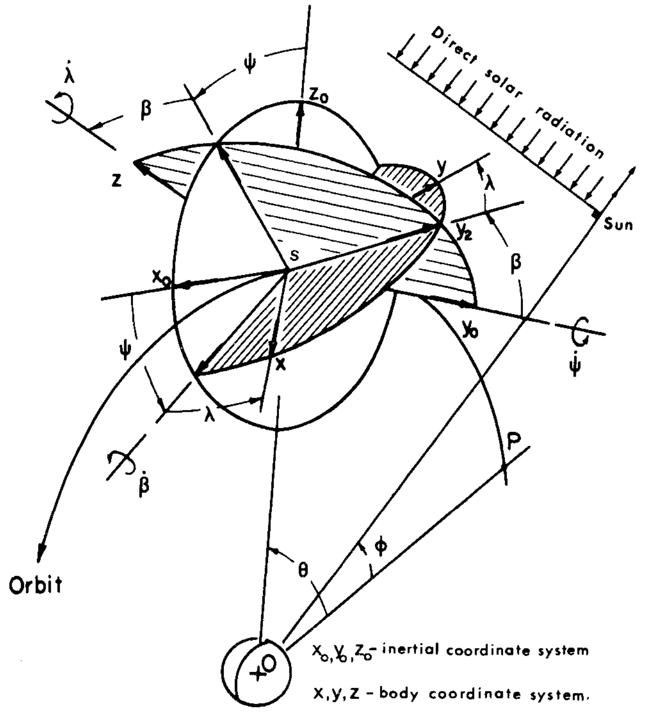

Fig. 1 Geometry of satellite motion.

Note that λ being a cyclic coordinate, its conjugate momentum represents a constant of the motion, thus establishing the first integral in the yaw degree of freedom. Several points of interest become apparent:

i) λ motion is governed by ψ and β, and because of their periodic character λ motion is also going to be almost periodic, particularly for small amplitude librations normally encountered in practice;

ii) Equations (1a) and (1b) are independent of λ.

The objective, therefore, is to determine initial conditions leading to coupled periodic motion in pitch and roll.

Replacing trigonometric functions by their series expansions, neglecting the fifth and higher order terms in ψ, β, and their derivatives, and collecting the nonlinear terms on the right hand side, the equations can be written as[10]

$$\left.\begin{aligned} \psi''+n_1{}^2\psi &= f_1(\theta,\psi,\psi',\beta,\beta')+c_e\sin(\theta-\alpha) \\ \beta''+n_2{}^2\beta &= f_2(\theta,\psi,\psi',\beta,\beta') \end{aligned}\right\} \tag{2}$$

where:

$n_1{}^2 = 3K_i$

$n_2{}^2 = 3K_i+1$

$c_f = c(8+4g)(1+e)^3/3\pi$

$c_e = [(c_f\cos\phi+2e)^2+c_f{}^2\sin^2\phi]^{1/2}$

$\alpha = \tan^{-1}[c_f\sin\phi/(c_f\cos\phi+2e)]$

ϕ = solar aspect angle

g = aspect ratio of the satellite, depends on satellite dimensions and material characteristics[10]

c = solar parameter, a function of the satellite material, geometry, inertia and orbital characteristics[10]

$$\begin{aligned} f_1 = {} & 2\beta\beta'+2\beta\beta'\psi'+(\tfrac{2}{3})\beta'\beta^3+(\tfrac{2}{3})n_1{}^2\psi^3+2e\psi'\sin\theta-e^2\sin 2\theta- \\ & e^2\psi'\sin 2\theta-2e^3\cos^2\theta\sin\theta-2e^3\psi'\cos^2\theta\sin\theta- \\ & 2e^4\cos^3\theta\sin\theta+n_1{}^2e\psi\cos\theta-n_1{}^2e^2\psi\cos^2\theta+ \\ & n_1{}^2e^3\psi\cos^3\theta-(\tfrac{2}{3})n_1{}^2e\psi^3\cos\theta+(\tfrac{1}{2})c_f\beta^2\sin(\theta-\phi)- \\ & 4ec_f\cos\theta\sin(\theta-\phi)-2ec_f\beta^2\cos\theta\sin(\theta-\phi)- \\ & 10e^2c_f\cos^2\theta\sin\theta-20e^3c_f\cos^3\theta\sin(\theta-\phi) \\ f_2 = {} & -2\beta\psi'-\psi'^2\beta+(\tfrac{4}{3})\psi'\beta^3+n_1{}^2\psi^2\beta+(\tfrac{2}{3})n_2{}^2\beta^3+ \\ & 2e\beta'\sin\theta-e^2\beta'\sin 2\theta+e^3\beta'\sin\theta\cos^2\theta+n_1{}^2e\beta\cos\theta- \\ & n_1{}^2e^2\beta\cos^2\theta+n_1{}^2e^3\beta\cos^3\theta-n_1{}^2e\beta\psi^2\cos\theta-(\tfrac{2}{3})n_1{}^2e\beta^3\cos\theta \end{aligned}$$

For small amplitude motion, each term in f_1 and f_2 is comparatively small, hence the method of variation of parameters is applicable. Accordingly, a solution is considered in the form

$$\left.\begin{aligned} \psi &= a_1(\theta)\sin\{n_1\theta+\delta_1(\theta)\}+\{c_e/(n_1{}^2-1)\}\sin(\theta-\alpha) \\ \beta &= a_2(\theta)\sin\{n_2\theta+\delta_2(\theta)\} \end{aligned}\right\} \tag{3}$$

where the amplitudes a_1, a_2 and phase angles δ_1, δ_2 are slowly varying parameters determined from the following average relations:

$$\left.\begin{aligned} a_i' &= (1/8\pi^3 n_i)\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f_i^*\cos\varepsilon_i\, d\varepsilon_1\, d\varepsilon_2\, d\theta \\ \delta_i' &= -(1/8\pi^3 n_i a_i)\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f_i^*\sin\varepsilon_i\, d\varepsilon_1\, d\varepsilon_2\, d\theta \end{aligned}\right\} i=1,2$$

Here

$$\varepsilon_i = n_i\theta+\delta_i(\theta)$$

$$f_i^* = f_i(\theta, a_1\sin\varepsilon_1, a_1n_1\cos\varepsilon_1, a_2\sin\varepsilon_2, a_2n_2\cos\varepsilon_2)$$

Using the conditions of orthogonality while evaluating the above integrals leads to the final solution which can be conveniently written in the following form:

$$\psi = D_1\cos(K_1\theta)+D_2\sin(K_1\theta)+\{c_e/(n_1{}^2-1)\}\sin(\theta-\alpha) \tag{4a}$$

$$\beta = D_3\cos(K_2\theta)+D_4\sin(K_2\theta) \tag{4b}$$

where

$$a_1{}^2 = D_1{}^2+D_2{}^2$$

$$a_2{}^2 = D_3{}^2+D_4{}^2$$

$$K_1 = n_1[1-\{(a_1{}^2-e^2)/4\}-c_e{}^2/\{2(n_1{}^2-1)^2\}]$$

$$K_2 = n_2[1-(a_2{}^2/4)-c_e{}^2/\{4n_2{}^2(n_1{}^2-1)\}+\{n_1{}^2/4n_2{}^2\}e^2]$$

and D_1, D_2, D_3 and D_4 are obtained using the initial conditions:

$$\begin{aligned} D_1 = {} & [\psi_0-\{c_e/(n_1{}^2-1)\}\sin(\theta_0-\alpha)]\cos(K_1\theta_0)- \\ & (1/K_1)[\psi_0'-\{c_e/(n_1{}^2-1)\}\cos(\theta_0-\alpha)]\sin(K_1\theta_0) \\ D_2 = {} & [\psi_0-\{c_e/(n_1{}^2-1)\}\sin(\theta_0-\alpha)]\sin(K_1\theta_0)+ \\ & (1/K_1)[\psi_0'-\{c_e/(n_1{}^2-1)\}\cos(\theta_0-\alpha)]\cos(K_1\theta_0) \\ D_3 = {} & \beta_0\cos(K_2\theta_0)-(1/K_2)\sin(K_2\theta_0) \\ D_4 = {} & \beta_0\sin(K_2\theta_0)+(1/K_2)\cos(K_2\theta_0) \end{aligned}$$

3. Results and Discussion

This approximate solution provides a considerable insight into the system behavior. Here, our interest is confined to the periodic motion. It is apparent from the approximate expression (Eq. 4b) that out-of-plane librations executed by the satellite are always periodic. Furthermore, the period of this roll motion,

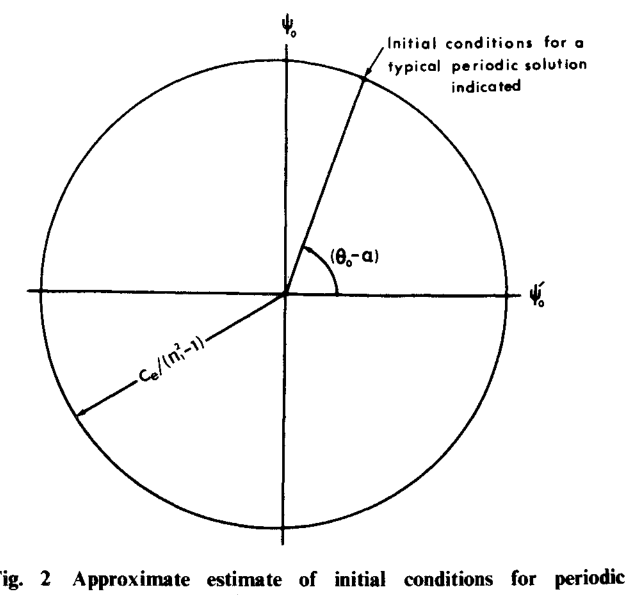

Fig. 2 Approximate estimate of initial conditions for periodic (period = 2π) librations.

which exhibits weak amplitude dependence, is $\simeq(1/n_2)$(orbital time). Thus, for a given satellite, the frequency of out-of-plane motion is fixed. On the other hand, pitching motion presents several interesting possibilities:

i) For satellites in circular orbits and with negligible solar radiation effect (i.e., $c_e = 0$), the motion of periodicity $\simeq (1/n_1)$ (orbital period) is assured.

ii) For initial conditions satisfying the relations

$$\left.\begin{aligned}\psi_0 &= \{c_e/(n_1^2-1)\}\sin(\theta_0-\alpha)\\ \psi_0' &= \{c_e/(n_1^2-1)\}\cos(\theta_0-\alpha)\end{aligned}\right\} \tag{5}$$

$D_1 = D_2 = 0$ and the pitching motion of orbital period ensues. The parameter $c_e/(n_1^2-1)$ appears to be quite significant as it governs not only the initial conditions leading to the periodic motion but also its amplitude. Thus, the initial conditions for the periodic motion are affected by solar radiation pressure in addition to orbital eccentricity and the satellite mass distribution. Interestingly, they can be represented quite simply by a circle as shown in Fig. 2.

iii) In general, when all the three terms are present, the motion is still periodic. However, the periodicity is now governed by K_1, which can be approximated as a rational number, m/n. As dictated by the third term, the lowest period is of course 2π, when $K_1 = m/n = 1$, the largest value for K_1 being one. Otherwise, the period would simply correspond to n orbits.

To assess accuracy of the above procedure, librational response of the system was obtained through numerical integration of the exact equations of motion (1a, 1b) using the initial conditions as given by Eq. (5) and compared with that predicted by the approximate solution of Eq. (4). Typical comparisons are presented in Fig. 3. The correlation is indeed satisfactory. Thus, the method represents a rather simple and effective procedure for determining periodic solutions of such a complex system. Its usefulness during initial stages of satellite design is apparent.

A comment concerning the applicability of this procedure to more complex situations involving spacecraft flexibility and/or damping would be appropriate. Periodic solutions continue to be important[11] as their spinal character with respect to the stability region is maintained.[12] Of course, routine algebraic manipulations would increase because of the rather involved character of integrand f_i^*. However, the method should provide a quick estimate of the periodic solutions, even for such formidable systems.

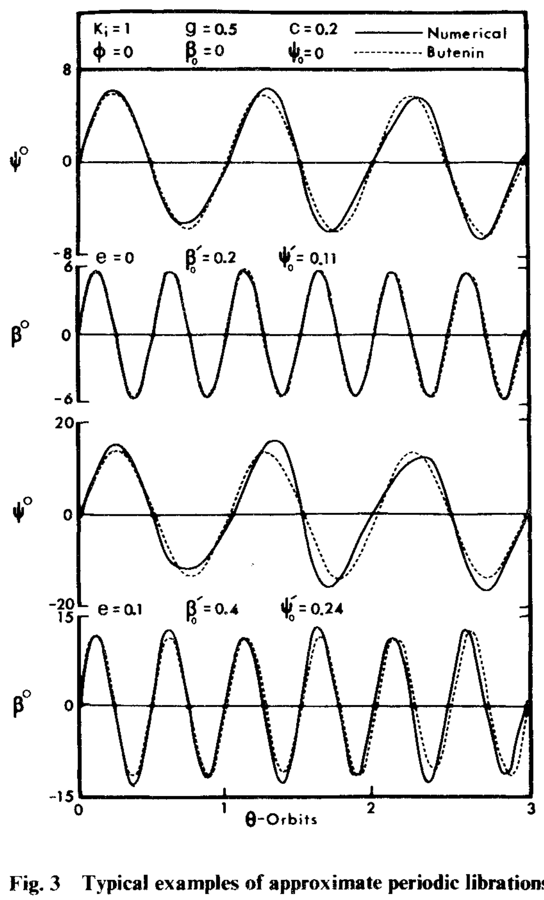

Fig. 3 Typical examples of approximate periodic librations.

References

[1] Zlatousov, V. A., Okhotsimsky, D. E., Sarychev, V. A., and Tozhevsky, A. P., "Investigation of Satellite Oscillations in the Plane of an Elliptic Orbit," *Proceedings of the XIth International Congress of Applied Mechanics*, Springer-Verlag, Berlin, 1964, pp. 436–439.

[2] Modi, V. J. and Brereton, R. C., "Periodic Solutions Associated with the Gravity-Gradient-Oriented System; Part I: Analytical and Numerical Determination," *AIAA Journal*, Vol. 7, No. 7, July 1969, pp. 1217–1225.

[3] Modi, V. J. and Brereton, R. C., "Periodic Solutions Associated with the Gravity-Gradient-Oriented System; Part II: Stability Analysis," *AIAA Journal*, Vol. 7, No. 8, Aug. 1969, pp. 1465–1468.

[4] Modi, V. J. and Shrivastava, S. K., "On the Limiting Regular Stability and Periodic Solutions of a Gravity Oriented System in the Presence of the Atmosphere," *CASI Transactions*, Vol. 5, No. 1, March 1972, pp. 5–10.

[5] Modi, V. J. and Neilson, J. E., "On the Periodic Solutions of Slowly Spinning Gravity Gradient System," *Celestial Mechanics—An International Journal of Space Dynamics*, Vol. 5, No. 2, May 1972, pp. 126–143.

[6] Krylov, N. M. and Bogoliubov, N. N., *Introduction to Nonlinear Mechanics*, Princeton University Press, Princeton, N.J., 1943.

[7] Butenin, N. V., *Elements of Nonlinear Oscillation*, Blaisdell, New York, 1965.

[8] Modi, V. J. and Shrivastava, S. K., "Coupled Librational Motion of an Axi-Symmetric Satellite in a Circular Orbit," *The Aeronautical Journal of the Royal Aeronautical Society*, Vol. 73, No. 704, Aug. 1969, pp. 674–680.

[9] Modi, V. J. and Shrivastava, S. K., "Librations of Gravity-Oriented Satellites in Elliptic Orbits through Atmosphere," *AIAA Journal*, Vol. 9, No. 11, Nov. 1971, pp. 2208–2216.

[10] Kumar, K., "Effect of Solar Radiations on the Attitude Dynamics of Gravity Oriented Satellites," Ph.D. thesis, May 1972, The University of British Columbia, Vancouver, British Columbia, Canada.

[11] Modi, V. J. and Brereton, R. C., "The Planar Motion of a Damped Satellite," *Transactions of the Canadian Aeronautics and Space Institute*, Vol. 2, No. 1, March 1969, pp. 44–48.

[12] Modi, V. J. and Kumar, K., "Librational Dynamics of Satellites with Thermally Flexed Appendages," Paper presented at the *AAS/AIAA Astrodynamics Specialists Conference*, Vail, Colorado, July 1973, to be published.

Studies on Mixed Fuels—Hydrazine and Ethyl Alcohol System

R. P. Rastogi,* H. J. Singh,† and K. Kishore‡
University of Gorakhpur, Gorakhpur (U.P.), India

Introduction

ALTHOUGH hydrazine has long been known to be an attractive rocket fuel, several problems arise in its use. Hydrazine readily undergoes catalytic decomposition on many metal surfaces. In the vapor phase it has a strong tendency towards explosive decomposition. Hence for storing hydrazine

Received April 26, 1973; revision received September 10, 1973. We would like to thank the Ministry of Defence, Government of India for financial support.

Index categories: Fuels and Propellants, Properties of; LV/M Fuel and Propellant Systems (Including Storage and Transfer).

* Professor, Department of Chemistry. Associate Fellow AIAA.

† Senior Research Fellow, Department of Chemistry.

‡ Lecturer, Department of Chemistry.